CALABI–YAU EMBEDDING OF SOME REDUCIBLE SURFACES VIA ORBIFOLDS

1. Introduction

The purpose of this note is to introduce the local Gromov–Witten invariants of a local del Pezzo orbifold and to prove embeddability of some higher rank surfaces considered in [6]. Here, a del Pezzo orbifold is a 2 dimensional orbifold constructed in the following way. Given a smooth projective surface S, a normal crossing divisor $D = D_1 + \dots D_n$, and a vector $\vec{r} \in \mathbb{Z}^n$, we can attach a stabilizer group of μ_{r_i} to D_i, and construct a smooth DM stack $\mathcal{S}_{D,\vec{r}}$. It is called a root stack of S along D [2]. These are examples of cyclotomic stacks in the sense of [?AH].

More explicitly, recall that the classifying stack $B\mathbb{G}_m = [\mathrm{Spec}\ \mathbb{C}/\mathbb{G}_m]$ classifies $\mathbb{G}_m$-bundles (and thus line bundles). Simirlarly, the stack $[\mathbb{A}^1/\mathbb{G}_m]$ classifies line bundle and a section. In the following diagram, the map π denote a line bundle corresponding to a map f and a data of a map $X \to [\mathbb{A}^1/\mathbb{G}_m]$ is equivalent to a section of π and a choice of f. Hence this determines a line bundle and a global section. Similarly, a map to $[\mathbb{A}^r \to \mathbb{G}_m^r]$ corresponds to r line bundles and r global sections.

$$\begin{tikzcd} X \times_{\mathbb{G}_m} [\mathbb{A}^1/\mathbb{G}_m] \arrow[r] \arrow[d, "\pi"] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d] \\ X \arrow[r, "f"] & B\mathbb{G}_m \end{tikzcd}$$

Given a r_i-th power map θ_{r_i} that sends (L_i, s_i) to (L_i^r, s_i^r) where s_i is the section whose zero scheme is D_i, the root stack along D with order $\vec{r}$ is done by taking the following fiber product

$$S_{D,\vec{r}} = S_{\vec{L_i},\vec{s},\vec{r}} = S \times_{[\mathbb{A}^n/\mathbb{G}_m^n],\theta_{\vec{r}}} [\mathbb{A}^n/\mathbb{G}_m^n]. \tag{1.1}$$

For our application, we will only concern when $n = 1$, $r = 2$ so we write simply $\mathcal{S}$ for $\mathcal{S}_{D,\vec{r}}$. We work over $\mathbb{C}$.

Remark 1.2. These constructions all make sense for general algebraic stakcs. However, we won't need such generalities in this note. See Cadman's papers for more details.

2. Geometry of orbifolds

Here we recall a bit of well known facts on orbifolds. A good reference is [4], chapter 4. See also [2] on root stacks. The root stack construction comes with a map c to its coarse moduli space, which is just the underlying projective surface S

$$c : \mathcal{S} \to S.$$

It is an isomorphism on $S \setminus D$ and thus the generic stabilizer is trivial. Hence, it is a Gorenstein orbifold. Its inertia stack is given by the disjoint union

$$I\mathcal{S} = \mathcal{S} \sqcup D.$$

It carries a canonical line bundle $\omega_{\mathcal{S}}$ or canonical divisor $K_{\mathcal{S}}$. The orbi-line bundle $\omega_{\mathcal{S}}$ is given by

$$\omega_{\mathcal{S}} = c^*\omega_S \otimes \mathcal{L}$$

where $\mathcal{L}^{\otimes 2} \simeq \mathcal{O}_S(D)$ is an orbi-line bundle which is *not* a pullback of a usual line bundle on S. Let (z_1, z_2) be local coordinates with D given by $z_1 = 0$ and let (x_1, x_2) be a corresponding local coodrinate system of the upstair orbifold. Then we have

$$c^*(dz_1 \wedge dz_2) = 2x_1 dx_1 \wedge dx_2$$

and the ramification divisor is given by $x_1 = 0$. This shows that orbifold line bundle $\omega_{\mathcal{S}}$ is *not locally trivial* around points on D and it has the induced μ_2 action from the downstair. On $S \setminus D$, it is just the usual line bundle.

The notion of Cartier divisor can be adapted to the case of orbifolds. Intuitively, they are given by local equations by an *G-invariant* functions, where G is the local stabilizer group. An integral divisor D on S, when lifted on $\mathcal{S}$, can be divisible even if D is not divisible on S. By the same way as the canonical line bundle, the canonical divisor is given by

$$K_{\mathcal{S}} = c^*K_S + \frac{1}{2}D$$

which is a $\mathbb{Q}$-divisor. The divisor $\frac{1}{2}D$ should be understood as a corresponding divisor to $\mathcal{L}$ above. The anticanonical orbibundle $K_{\mathcal{S}}^{-1}$ can also be defined.

The notion of Chern class also generalizes, having value in $H^i(X, \mathbb{Q})$.

We say a $\mathbb{Q}$-Cartier divisor D is ample if it is locally ample and some positive multiple of it (viewed as a Cartier divisor on S) is an integral ample Cartier divisor.

Definition 2.1. Let $\mathcal{S}$ be an orbifold constructed from the root stack construction on (S, D). If $K_{\mathcal{S}}^{-1}$ is an ample divisor, then we say $\mathcal{S}$ is a del Pezzo orbifold.

3. Calabi–Yau embedding of some reducible surfaces

Let $\mathcal{X}$ be the total space of $K_{\mathcal{S}}$ on $\mathcal{S}$ which is again an orbifold. It is a Calabi-Yau orbifold of dimension 3 which we call an local (orbi)surface. Its coarse moduli space X is known to be $\mathbb{Q}$-factorial and it has a singularity along D in the zero section and there exists a crepant resolution by a single blow-up. Locally, the coarse moduli space look like

$$\text{Spec } \mathbb{C}[y_1, y_2, y_3]^{\mu_2} \tag{3.1}$$

where (-1) acts on y_1 and y_3 via multiplication by (-1). Hence, it is transverse A_1 singularity. Since blowup commute with flat base change and the resolution of A_1 singularity consists of a single $\mathbb{P}^1$, after the resolution we get $\mathbb{P}^1$ fibration over D. **Can it be other things? Understand $\mathbb{F}_2^3$ using this** Resolving this singularity, we get a genuine (non-stacky, non-compact) Calabi–Yau 3-fold X which has $S \cup_D R$ where R is a ruled surface over D.

Theorem 3.2. *Let S be a smooth projective surface and D be a smooth divisor with self intersection $D^2 = d$ and arithmetic genus (viewed as a smooth curve) g. Then the above construction gives a (non-compact) Calabi–Yau 3-fold containing $S \cup_D R$ where R is a ruled surface over D such that the self-intersection of D in R is $2g - 2 - d$.*

Proof. The local coordinate of a coarse moduli space is given by Spec $(\mathbb{C}[x,y,z]^{\mu_2})$ where μ_2 acts with multiplication by $(-1,1,-1)$ on (x,y,z). This is transverse A_1 singularity, locally

$$\text{an } A_1 \text{ singulairty } \times \mathbb{A}^1.$$

Its resolution is given by blowing up the fixed loci, which is $(0,y,0)$ because the resolution is given by resolution of A_1 singularity with a product of $\mathbb{A}^1$. Since blowup commute with flat morphism, it can be computed Zariski locally. Thus locally after the blowup the exceptional divisor is $\mathbb{P}^1 \times \mathbb{A}^1$. The induced map on the exceptional divisor R is $\pi|_R : R \to D$ is thus a $\mathbb{P}^1$-bundle. In other words, R is a geometrically ruled surface over D and thus it is a projectivization of a rank 2 vector bundle over D. □

In the case D is $\mathbb{P}^1$ and $D^2 \geq -1$, the Calabi–Yau condition forces that we get a unique Hirzebruch surface.

Theorem 3.3. *Let S be a smooth projective surface and D be a smooth rational curve with self-intersection $D^2 = d \geq -1$. Then there is a non-compact Calabi-Yau 3-fold X containing $S \cup \mathbb{F}_{-2-d}$ where $S \cap \mathbb{F}_{-2-d} = D$ is the unique negative curve in $\mathbb{F}_{-2-d}$.*

Proof. If $D \simeq \mathbb{P}^1$ and $D^2 \geq -1$, then the self-intersection of D in R is also negative as their sum must be -2. By the previous theorem, R is ruled over D, it must be a Hirzebruch surface which can only have a unique section with negative self-intersection. □

Example 3.4. Let $S = \mathbb{P}^2$ and D be a smooth conic. In this case, the root stack $\mathcal{S}$ is actually a global quotient stack $\mathcal{S} = [\mathbb{F}_0/\langle i \rangle]$ where $p : \mathbb{F}_0 \to \mathbb{P}^2$ is the double cover of $\mathbb{P}^2$ branched along D and $i : \mathbb{F}_0 \to \mathbb{F}_0$ is the involution that comes with the double cover. Although scheme-theoretic quotient is just $\mathbb{P}^2$, it has nontrivial stabilizer attached to D. Consider the toal space of line bundle $\omega_{\mathcal{S}}$ on $\mathcal{S}$. Outside D, it is just the usual line bundle. Thus, the orbifold structure is supported over D. In local coordinate, the stabilizer just acts as multiplication by (-1) on the fiber and the defining equation. Thus, the coarse moduli space of $\mathcal{X}$ has a transverse A_1 singularity. Being Gorenstein Calabi–Yau, it admits a crepant resolution which is just the blowing up the singular locus in this case. The resulting blow-up is $\mathbb{F}_6$ by the Calabi–Yau condition and we get a normal crossing divisor $\mathbb{P}^2 \cup \mathbb{F}_6$.

This example can be constructed from $\mathrm{Tot}K_{\mathbb{F}_0}/\langle \tau \rangle$ where τ is acting on the fiber as well.

Hence the invariant for $\mathbb{P}^2 \cup \mathbb{F}_6$ in [6] could have been defined using this Calabi–Yau embedding.

Example 3.5. Let $S = \mathbb{P}^2$ and $D = \ell$ be a line. In this case, there does not exist a double cover of $\mathbb{P}^2$ branched along a line, so it cannot be constructed as a global quotient of a variety. However, there is such a double cover locally, and it glues in the category of orbifolds. Let $\mathcal{S}$ be such an orbifold, and $\mathcal{X} = \mathrm{Tot}(\omega_{\mathcal{S}})$. The coarse moduli space X has transverse A_1 singularity along a line. Blowing up this line, we get a divisor $\mathbb{P}^2 \cup \mathbb{F}_3$ in the resolution $\widetilde{X}$.

Since a coordinate axis in $\mathbb{P}^2$ is a torus invariant divisor, the orbifold $\mathcal{X}$ has toric description, as a partial resolution of isolated quotient singularity $\mathbb{C}^3/\mu_5$.

Example 3.6. Let $S = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and D be a smooth curve of class $f_1 + f_2$, where f_i is a ruling of $\mathbb{F}_0$. It is not a torus invariant divisor, and there is no double cover in branched along D in the categories of varieties. Let $\mathcal{S}$ be the (square) root stack along D and let $\mathcal{X}$ be the total space of canonical bundle on $\mathcal{S}$. Blowing up D in the zero section, we get $\mathbb{F}_0 \cup \mathbb{F}_4$.

Example 3.7. It can be extended to root stack of higher order of stabilizer group. In this case, the canonical bundle becomes $K_{\mathcal{S}} = c^* K_S + \left(1 - \frac{1}{n}\right) D$ and thus the zero section has A_{n-1} singularity. we need more than one blowups to resolve the singularity, and we get more than one ruled surface attached to S. For instance, the partial resolution of $\mathbb{C}^3/\mu_{2k+1}$ can be obtained from this orbifold construction, by considering $\mathbb{P}^2_{\ell,k}$

In particular, we have the following.

Proposition 3.8. *The snc del Pezzo surfaces in* [6] *can all be embedded into a non-compact Calabi–Yau 3-fold.*

Proof. The only remaining case is $\mathbb{F}_0 \cup \mathbb{F}_4$. Since the gluing curve is of class $f_1 + f_2$ which has self-intersection 2 in $\mathbb{F}_0$, Theorem 3.2. produces an embedding. □

We expect to produce all Calabi–Yau 3-fold containing shrinkable surfaces. However, we do not yet know how to deal with surfaces with some points on a component blown-up.

References

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